

CCFU Proof 28

$$\text{Stab}(\Omega_W) \cong \mathfrak{g}_2(\text{split})$$

Given. Ω_W from Proof 22. $\dim \text{Stab}(\Omega_W) = 14$ [Proof 23].

G_2 identification (CCFU Script 2):

1. $\dim \mathfrak{g} = 14$.
2. \mathfrak{g} closed under commutators (Lie subalgebra).
3. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (perfect).
4. Killing form nondegenerate (semisimple — Cartan criterion).
5. \mathbb{R}^7 irreducible (commutant = scalars — Schur).
6. Simple (ad-commutant dim 1).
7. Killing signature (8, 6) (split real form — Cartan 1914).
8. $\mathfrak{g} \subset \mathfrak{so}(b_{\Omega_W})$ (preserves Hitchin metric).

By Killing's classification (1887): the only 14-dimensional simple Lie algebra (over \mathbb{C}) is \mathfrak{g}_2 .

By Cartan's classification of real forms (1914): Killing signature (8, 6) identifies the split real form.

Therefore $\text{Stab}(\Omega_W)$ has Lie algebra $\mathfrak{g}_2(\text{split})$. ■

External references:

Killing 1887 (classification of simple Lie algebras over \mathbb{C}).

Cartan 1914 (classification of real forms).

Cartan criterion (Killing nondegenerate \Leftrightarrow semisimple).

Schur's lemma (commutant = scalars \Leftrightarrow irreducible).

[Dependencies: Proofs 22, 23.]